potential φ. The component due to the piezoelectric effect and that also has an infinite propagation velocity, is not contained explicitly in the sum $\Psi^{(m)}$ and is determined after solving problem (1.2)-(1.5).

In conclusion, we note that the method described enables the problem to be solved for more complex electrical boundary conditions as compared with (1.4). In place of the values of the potentials on the electrodes in considering applied problems, the magnitudes are often given for the currents through them or the characteristics of the outer electrical loops. In these cases, unknown values of the potentials are introduced into the boundary conditions (1.4) and are then determined from the equations for the currents in the outer loops or the charge conservation conditions.

REFERENCES

1. PAO Y.-H., Elastic waves in solids, Trans. ASME. J. Appl. Mech., 50, 4b, 1983.
2. ERINGEN A.C. and SUHUBI E.S., Elastodynamics, 2, Linear theory, Academic Press, New York, 1975.
3. WEAVER R.L. and PAO Y.-H., Axisymmetric elastic waves excited by a point source in a plate, Trans. ASME, J. Appl. Mech., 49, 4, 1982.
4. ZHARII O.YU., Regularities of the generation of Rayleigh waves during non-stationary vibrations of a half-plane, Akust. Zh., 34, 4, 1988.
5. PARTON V.Z. and KUDRYAVTSEV B.A., Electromagnetoelasticity of Piezoelectric and Electrically Conducting Bodies. Nauka, Moscow, 1988.
6. ZHARII O.YU. and ULITKO A.F., Introduction to the Mechanics of Non-statioary Vibrations and Waves, Vishcha Shkola, Kiev, 1989.
7. ZHARII O.YU., Wave processes in piezoceramic pulse generators under electrical discharge, The Strength o Polycrystalline Ferroelectrics, A.F. Ioffe Inst. of Solid State Physics, Leningrad, 1981.
8. ZHARII O.YU. and ULITKO A.F., Electrical discharge of a piezoceramic rod under stationary mechanical excitation, Mathematical Methods and Physicomechanical Fields, 13, Naukova Dumka, Kiev, 1981.
9. FIKHTENGOL'TS G.M., Course in Differential and Integral Calculus, 3, Nauka, Moscow, 1969.

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HYPERSINGULAR INTEGRALS IN PLANE PROBLEMS OF THE THEORY OF ELASTICITY*

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This paper is devoted to the solution of plane problems of the theory of elasticity by the method of discontinuous displacement using finite-part integrals (FPI). Two different integral equations (a real one and a complex one) with FPI's are obtained for the plane of a body with cracks. This opens the way for using arbitrary approximations of displacement discontinuities. The article contains integral formulae for FPI's used in the approximation of displacement discontinuities by polynomials of any order for internal elements and by special functions accounting for the asymptotic behaviour for the boundary elements. Therefore, prerequisites for increasing the accuracy of computations are created. The results of numerical experiments carried out indicate that there is a sharp increase (by two orders of magnitude) in the accuracy of the solution of the crack problem in which the integral formulae in question are used.

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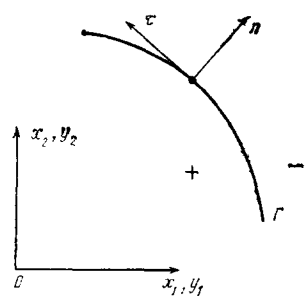
During the last decade finite-part integrals (FPI), which were introduced by Hadamard /1/, have become a useful tool in solving problems in the mechanics of continuous media /2-8/, particularly in cases where the functions being sought undergo discontinuities on some surfaces (lines). For the three-dimensional problem of a crack in a rigid body, the high effectiveness of FPI's was shown in /4, 8/. FPI's are also useful in plane crack problems. It suffices to mention that the method of displacement discontinuities /9/, which is widely used in computations, consists essentially in applying FPI's to a simple piecewise constant approximation of the crack and the displacement discontinuity along the crack. However, the integral equation solved by the approximation was not derived in /9/. This made it difficult to use more accurate approximations. Only recently have some articles appeared in which displacement discontinuities are expressed in terms of integral formulae /10, 11/. It has been proved that the accuracy increases substantially in this case (sometimes by an order of magnitude). However, the question concerning arbitrary approximations, and, in particular, approximations in which the asymptotic behaviour of displacement discontinuities at the ends of the crack is taken into account, has remained open. This has become all the more pressing since a rigorous proof of the applicability of FPI's to the boundary elements of a rectilinear crack was given in /12/.

1. We consider an unbounded domain with a cut along a line Γ (see the figure). We fix a direction of motion along Γ and we choose a normal vector $\mathbf{n}$ pointing to the right-hand side of the direction. We also adopt the convention that the limiting values of a function as the argument approaches Γ from outside (inside) with respect to the normal vector will be marked by a plus (minus) sign.



Suppose that there are forces of the same magnitude but in opposite directions acting on the edges of the cut. In this case, if the normal vector is fixed, then the vector $\boldsymbol{\sigma}_{\mathbf{n}}$ representing the forces is continuous and equal to the given vector $\boldsymbol{\sigma}_{\mathbf{n}0}$ ($\sigma_{\mathbf{n}}^+ = \sigma_{\mathbf{n}}^- = \sigma_{\mathbf{n}0}$ on Γ).

At an internal point of Γ the displacement vector can be represented by a double-layer potential

$$u_i(\mathbf{x}) = -\int_\Gamma W_{ij}(\mathbf{x}, \mathbf{y}) \Delta u_j(\mathbf{y}) \, d_{\mathbf{y}}\Gamma, \quad \mathbf{x} \overline{\in} \Gamma \tag{1.1}$$

For an isotropic body

$$W_{ij}(\mathbf{x}, \mathbf{y}) = -\frac{1}{2\pi}\left\{\left[\frac{\nu}{1-\nu} n_j(\mathbf{y}) \frac{\partial}{\partial x_i} + n_i(\mathbf{y}) \frac{\partial}{\partial x_j} + \delta_{ij} \frac{\partial}{\partial \mathbf{n}}\right] \ln \frac{1}{R} - \right. \tag{1.2}$$
$$\left. \frac{1}{8(1-\nu)} \frac{\partial^3}{\partial x_i \partial x_j \partial \mathbf{n}} (R^2 - 2R^2 \ln R) \right.$$
$$R^2 = (x_i - y_i)^2, \quad \partial/\partial \mathbf{n} = n_k \partial/\partial x_k$$

where ν is Poisson's ratio, $\Delta\mathbf{u}$ is the potential density vector, and δ_{ij} is the Kronecker delta. It is understood that summation is carried out whenever a Latin index is repeated twice. The minus sign is written in front of the integral in (1.1) in order that some special cases of the expressions that follow are identical with the formulae that are traditionally used. All subsequent considerations remain valid also for the three-dimensional case when Γ is a surface. The only difference is that $\ln(1/R)$ in (1.1) is replaced by $1/(2R)$ and the function $R^2 - 2R^2 \ln R$ is replaced by $2R$ /5, 7, 8, 13/.

By virtue of the properties of the double-layer potential /14/, $\Delta\mathbf{u}$ represents the discontinuity of the displacement vector on Γ: $\Delta\mathbf{u} = \mathbf{u}^+ - \mathbf{u}^-$. The displacements $\mathbf{u}(\mathbf{x})$ outside Γ satisfy all equations of the theory of elasticity. From these equations we find the stresses $\sigma_{ij}(\mathbf{x})$ and the forces $\sigma_{\mathbf{n}i}(\mathbf{x})$ acting on a unit surface element with the normal vector $\mathbf{n}(\mathbf{x})$:

$$\sigma_{ij}(\mathbf{x}) = -\int_\Gamma D_{ijk}(\mathbf{x}, \mathbf{y}) \Delta u_k(\mathbf{y}) \, d_{\mathbf{y}}\Gamma, \quad \mathbf{x} \overline{\in} \Gamma \tag{1.3}$$

$$\sigma_{\mathbf{n}i}(\mathbf{x}) = -\int_\Gamma F_{\mathbf{n}ik}(\mathbf{x}, \mathbf{y}) \Delta u_k(\mathbf{y}) \, d_{\mathbf{y}}\Gamma, \quad \mathbf{x} \overline{\in} \Gamma \tag{1.4}$$

where for an isotropic body (E is the modulus of elasticity)

$$D_{ijk}(\mathbf{x}, \mathbf{y}) = \frac{E\nu}{(1+\nu)(1-2\nu)} \delta_{ij} \frac{\partial W_{sk}}{\partial x_s} + \frac{E}{2(1+\nu)}\left(\frac{\partial W_{ik}}{\partial x_j} + \frac{\partial W_{jk}}{\partial x_i}\right) =$$
$$-\frac{E}{2\pi(1-\nu^2)}\left\{\left[\nu\delta_{ij} \frac{\partial^2}{\partial x_k \partial \mathbf{n}} + \nu n_k(\mathbf{y}) \frac{\partial^2}{\partial x_i \partial x_j} + \frac{1-\nu}{2}\left(n_i(\mathbf{y}) \frac{\partial}{\partial x_j} + n_j(\mathbf{y}) \frac{\partial}{\partial x_i}\right) \frac{\partial}{\partial x_k} + \right.\right.$$

$$\frac{1-\nu}{2}\left(\delta_{ik}\frac{\partial}{\partial x_j}+\delta_{jk}\frac{\partial}{\partial x_i}\right)\frac{\partial}{\partial \mathbf{n}}\Big]\ln\frac{1}{R}-\frac{1}{8}\frac{\partial^4}{\partial x_k\partial x_i\partial x_j\partial \mathbf{n}}(R^2-2R^2\ln R)\Big\}$$

$$F_{\mathbf{n}ik}(\mathbf{x},\mathbf{y})=D_{ijk}(\mathbf{x},\mathbf{y})\,n_j(\mathbf{x})=-\frac{E}{2\pi(1-\nu^2)}\Big\{\Big[\nu n_i(\mathbf{x})\frac{\partial^2}{\partial x_k\partial \mathbf{n}}+$$

$$\nu n_j(\mathbf{x})\,n_k(\mathbf{y})\frac{\partial^2}{\partial x_i\partial x_j}+\frac{1-\nu}{2}n_j(\mathbf{x})\left(n_i(\mathbf{y})\frac{\partial}{\partial x_j}+n_j(\mathbf{y})\frac{\partial}{\partial x_i}\right)\frac{\partial}{\partial x_k}+$$

$$\frac{1-\nu}{2}n_j(\mathbf{x})\left(\delta_{ik}\frac{\partial}{\partial x_j}+\delta_{jk}\frac{\partial}{\partial x_i}\right)\frac{\partial}{\partial \mathbf{n}}\Big]\ln\frac{1}{R}-$$

$$-\frac{1}{8}n_j(\mathbf{x})\frac{\partial^4}{\partial x_k\partial x_i\partial x_j\partial \mathbf{n}}(R^2-2R^2\ln R)\Big\}$$

(Relation (1.2) is used).

Potential (1.4) is continuous on Γ /13/. Thus, we only need to require that the limiting values of $\sigma_{\mathbf{n}}(\mathbf{x})$ should be equal to the vector $\sigma_{\mathbf{n}0}$ representing the forces given on Γ. As a result, we get the equation

$$\lim_{\mathbf{x}\to\mathbf{x}_0}\Big\{-\int_\Gamma F_{\mathbf{n}ik}(\mathbf{x},\mathbf{y})\,\Delta u_k(\mathbf{y})\,d_{\mathbf{y}}\Gamma\Big\}=\sigma_{\mathbf{n}i0}(\mathbf{x}_0),\quad \mathbf{x}_0\in\Gamma \tag{1.5}$$

which can also be written in the form

$$-\,\mathrm{v.\,f.}\int_\Gamma F_{\mathbf{n}ik}(\mathbf{x}_0,\mathbf{y})\,\Delta u_k(\mathbf{y})\,d_{\mathbf{y}}\Gamma=\sigma_{\mathbf{n}i0}(\mathbf{x}_0),\quad \mathbf{x}_0\in\Gamma \tag{1.6}$$

where the integral on the left-hand side is understood as a FPI. On the arc Γ we choose an interval Γ_0 containing $\mathbf{x}_0$. If an integral over Γ_0 of a term of the form $\Delta u_k(\mathbf{y})/R^2$ appears in the integral formulae, then the FPI can be evaluated by substituting the limits of integration representing the ends of Γ_0 into the corresponding formal expression. Otherwise, we subtract and add an expression of the form $\Delta u_k(\mathbf{x}_0)/R^2$ and we formally evaluate the integral of $1/R^2$ in which we substitute the limits of integration /3, 8/. In this case the integral containing the difference $\Delta u_k(\mathbf{y})-\Delta u_k(\mathbf{x}_0)$ is understood as the principal value in the Cauchy sense.

The proof that (1.5) and (1.6) are equivalent is carried out by repeating the proofs given in /2/ for the three-dimensional problem. Here the theory of pseudodifferential operators is used /15/. The result also remains valid in the space of functions with derivatives of class H^* (according to the classification given in /16/).

2. The other equation with FPI's can be obtained by using the limiting integral equation with complex variables /17/

$$\frac{1}{\pi i}\int_\Gamma\frac{w'(\tau)}{\tau-t}\,d\tau+k_3w'=f_1(t),\quad t\in\Gamma \tag{2.1}$$

$$t=x_1+ix_2,\quad \tau=y_1+iy_2,\quad f_1(t)=\sigma_{nn}^\circ+i\sigma_{n\tau}^\circ,\ k_3=k_4+k_5$$

$$k_4w'=-\frac{1}{2\pi i}\int_\Gamma w'(\tau)\frac{\partial}{\partial t}\ln\frac{\bar\tau-\bar t}{\tau-t}\,d\tau$$

$$k_5w'=\frac{1}{2\pi i}\int_\Gamma\overline{w'(\tau)}\frac{\partial\bar\tau}{\partial\tau}\frac{\partial}{\partial t}\frac{\tau-t}{\bar\tau-\bar t}\,d\tau$$

$$w'(t)=\frac{dw}{dt},\quad w=\frac{E}{4(1-\nu^2)}\Delta u,\quad \Delta u=\Delta u_1+i\Delta u_2$$

$$\Delta u_1=u_1^+-u_1^-,\quad \Delta u_2=u_2^+-u_2^-$$

A bar above a symbol denotes the complex conjugate. The normal vector $\mathbf{n}$ points to the right-hand side of the direction of Γ, $\boldsymbol{\tau}$ is a tangent vector pointing in this direction (see the figure), and σ_{nn}° and $\sigma_{n\tau}^\circ$ are the components of the vector representing the forces given on Γ.

The function $w'(t)$ (like $w(t)$) represents the difference between the limiting values of a function that is holomorphic in the complement of Γ. Thus /17/,

$$\frac{1}{\pi i}\int_\Gamma\frac{w'(\tau)}{\tau-t}\,d\tau=\frac{d}{dt}\left(\frac{1}{\pi i}\int_\Gamma\frac{w(\tau)}{\tau-t}\,d\tau\right)=\frac{1}{\pi i}\,\mathrm{v.\,f.}\int_\Gamma\frac{w(\tau)}{(\tau-t)^2}\,d\tau \tag{2.2}$$

where the last equality in the chain is proved in exactly the same way as in /3/.

Substituting (2.2) into (2.1) and using the fact that $k_3 w'$ can be integrated by parts, we get

$$\frac{1}{\pi i}\,\mathrm{v.\,f.}\int_\Gamma \frac{w(\tau)}{(\tau-t)^2}\,d\tau + k_6 w = \overset{\circ}{\sigma}_{nn} + i\overset{\circ}{\sigma}_{n\tau} \tag{2.3}$$

$$k_6 w = \frac{1}{2\pi i}\int_\Gamma w\,\frac{\partial^2}{\partial\tau\partial t}\ln\frac{\bar\tau-\bar t}{\tau-t}\,d\tau - \frac{1}{2\pi i}\int_\Gamma \bar w\,\frac{\partial^2}{\partial\tau\partial t}\,\frac{\tau-t}{\bar\tau-\bar t}\,d\tau$$

Finally, we obtain the complex equation with FPI's

$$\frac{E}{4(1-\nu^2)}\left\{\frac{1}{\pi i}\,\mathrm{v.\,f.}\int_\Gamma \frac{\Delta u_1 + i\Delta u_2}{(\tau-t)^2}\,d\tau + k_6(\Delta u_1 + i\Delta u_2)\right\} = \overset{\circ}{\sigma}_{nn} + i\overset{\circ}{\sigma}_{n\tau} \tag{2.4}$$

The advantage of this equation is that to solve it using computers, we can use an approximation by functions of one (complex) variable (for example, by Lagrange polynomials $L_n(z)$ with nodes at $z_k = x_{1k} + ix_{2k}$, where $k = 1,\dots,n$).

In the special case where Γ is a section $[a,b]$ of a straight line, we have $k_6 w = 0$. In this case, if the x_2-axis is parallel to the normal vector $\mathbf{n}$, then $\tau - t = x_1 - y_1$ and $d\tau = -dy_1$. Eq.(2.4) takes the form

$$(-1)^{k-1}\frac{E}{4\pi(1-\nu^2)}\,\mathrm{v.\,f.}\int_a^b \frac{\Delta u_k}{(x_1-y_1)^2}\,dy_1 = \overset{\circ}{\sigma}_{nk}, \quad k=1,2; \quad a<x_1<b \tag{2.5}$$

For a rectilinear crack with a normal discontinuity $(\sigma_{n1}^\circ = 0;\ k=2)$, Eq.(2.5) was obtained in /3/ (there is an insignificant difference consisting in that $f = -\Delta u_2/2$ and $p = -\sigma_{n2}^\circ$ are used in /3/ instead of Δu_2 and σ_{n2}°).

Compared with a singular equation of the type (2.1), Eq.(2.4) with FPI's has the useful feature that it involves exactly those mechanical quantities that are a matter of interest in connection with solutions of applied problems, namely the displacement discontinuities and the forces.

3. It follows from the results of Sect.1 that the integral formulae over a section of a stright line are useful for evaluating the FPI's in (1.6). Let us consider the integral over an interval $[a, b]$

$$J(x) = \mathrm{v.\,f.}\int_a^b \frac{f(\xi)}{(x-\xi)^2}\,d\xi \tag{3.1}$$

Let the values f_k of $f(x)$ be given at n points x_k belonging to this interval. The points x_k serve as the nodes of the integral formula. To approximate $f(x)$ it is convenient to use the functions

$$G_k(x_i) = \begin{cases} 1, & i=k \\ 0, & i\neq k \end{cases} \quad (i,k = 1,\dots,n)$$

Then

$$f(x) \approx \sum_{k=1}^{n} f_k G_k(x) \tag{3.2}$$

and substitution of (3.2) into (3.1) yields the integral formula

$$J(x) \approx \sum_{k=1}^{n} A_k(x) f_k, \quad A_k(x) = \mathrm{v.\,f.}\int_a^b \frac{G_k(\xi)}{(x-\xi)^2}\,d\xi \quad (k=1,\dots,n) \tag{3.3}$$

As the shape functions for the internal elements of Γ we can use the Lagrange polynomials

$$G_k(x) = L_k(x) = \prod_{i\neq k}^{n} \frac{x-x_i}{x_k-x_i} \tag{3.4}$$

Then, substituting (3.4) into (3.3), we get

$$A_k(x) = d^{-1}(M_{n-1} + a_1 M_{n-2} + \ldots + a_{n-1}M_0) \tag{3.5}$$

Here

$$M_s = \text{v. f.} \int_a^b \frac{\xi^s d\xi}{(x-\xi)^2} = s x^{s-1} \ln\left|\frac{x-b}{x-a}\right| +$$

$$\sum_{k=0,\, k\neq s-1}^{s} C_s^k (-1)^{s+1-k} x^k \frac{(x-b)^{s-k-1} - (x-a)^{s-k-1}}{s-k-1}, \qquad a_1 = -\sum_{m\neq k}^{n} x_m$$

$$a_2 = \sum_{m\neq j\neq k}^{n} x_m x_j, \ldots, a_{n-1} = (-1)^{n-1}\prod_{i\neq k}^{n} x_i, \quad d = \prod_{i\neq k}^{n}(x_k - x_i)$$

For elements adjacent to the ends of Γ it is best to use an approximation that takes into account the asymptotic behaviour of $f(x)$. Usually, the displacement discontinuity vanishes at the end-points and it tends to zero in such a way that it is proportional to $\sqrt{r}$ (where r is the distance from the end-points). Thus, we can assume that the shape functions for the elements adjacent to the ends of Γ are the following:

$$G_k(x) = \sqrt{(x-a)/(x_k-a)}\, L_k(x) \quad \text{(for the left end)} \tag{3.6}$$

$$G_k(x) = \sqrt{(b-x)/(b-x_k)}\, L_k(x) \quad \text{(for the right end)} \tag{3.7}$$

By substituting (3.6) and (3.7) into (3.3) and applying the integrals, we obtain the same formula as (3.5) except for the fact that d and the moments are replaced by

$$M_s = \sum_{k=2}^{s}\sum_{m=0}^{k-2} C_s^k C_{k-2}^m (-1)^{k+m} x^{s-k} (x-a)^{k-2-m} \frac{(b-a)^{m+3/2}}{m+3/2} + x^s B_0 + s x^{s-1} B_1,$$

$$B_0 = \frac{\sqrt{b-a}}{x-b} + \frac{1}{2\sqrt{x-a}}\Lambda, \quad B_1 = 2\sqrt{b-a} + \sqrt{x-a}\,\Lambda,$$

$$\Lambda = \ln\left|\frac{\sqrt{x-a}-\sqrt{b-a}}{\sqrt{x-a}+\sqrt{b-a}}\right|, \quad d = \sqrt{x_k-a}\prod_{i\neq k}^{n}(x_k - x_i)$$

for the left end, and by

$$M_s = \sum_{k=2}^{s}\sum_{m=0}^{k-2} C_s^k C_{k-2}^m (-1)^{k} x^{s-k} (x-b)^{k-2-m} \frac{(b-a)^{m+3/2}}{m+3/2} + x^s B_0 + s x^{s-1} B_1,$$

$$B_0 = \frac{\sqrt{b-a}}{a-x} + \frac{1}{2\sqrt{b-x}}\Lambda, \quad B_1 = -2\sqrt{b-a} - \sqrt{b-x}\,\Lambda,$$

$$\Lambda = \ln\left|\frac{\sqrt{b-x}-\sqrt{b-a}}{\sqrt{b-x}+\sqrt{b-a}}\right|, \quad d = \sqrt{b-x_k}\prod_{i\neq k}^{n}(x_k - x_i)$$

for the right end.

There is no need to give the numerical values of the weights A_k for specific values of n, x, a, and b, since the above analytic expressions are very simple, and direct computer calculations based on these expressions can be carried out more easily and with greater accuracy.

The given formulae with the real coordinate replaced by the complex coordinate $t = x_1 + ix_2$ can also be used to solve the complex equation with FPI's (2.4). Here even the above-mentioned advantages of the complex formulae manifest themselves: if (2.4) is used, it is not necessary to assume that an element of Γ is rectilinear or that it can be represented by a function of a section of a straight line, and it is sufficient to take the complex coordinates z_k of such an element of Γ as the nodes x_k.

4. The equations with FPI's and the integral formulae obtained in the present paper make it possible to increase the accuracy of computations of displacement discontinuities. As an illustration we present the results of numerical experiments carried out for the problem studied in /9/ concerning a rectilinear crack with constant internal pressure $\sigma_{n2}^\circ = -p$ ($\sigma_{n1}^\circ = 0$). The crack runs along the interval $[-1, 1]$.

The problem has an analytic solution /18/

$$\Delta u_2(x) = 4E^{-1}(1-\nu^2)\, p\sqrt{1-x^2} \tag{4.1}$$

which enables the accuracy of the solution of (2.4) to be checked.

In the case in question, Eq.(2.4) has the form (2.5) with $k = 2$. To obtain the solution of (2.5), partitions of the interval $[-1, 1]$ into various numbers of elements of the same length and various approximations within the elements were used. However, in order for the results to be comparable, the total number of nodal points was fixed to be either 15 or 30.

The approximating polynomials used within the elements were of the following order: a) zero $(n = 1)$, b) two $(n = 3)$, and c) four $(n = 5)$. Moreover, we used an approximation by polynomials of the second order for the internal elements combined with a term for the boundary elements in which the asymptotic behaviour was taken into account $(G_1 = \sqrt{1 \pm x}$, case d). Below we list the percentage errors of computations (with 15 nodal points) with respect to the exact solution (4.1) for each of the four cases:

x		0	0,1	0,3	0.4	0,5	0.7	0.8	0,9
	a	3,4	3,4	3.7	4,0	4.6	7.1	9,5	24
δ, %	b	1,8	1.8	1.9	2,1	2.4	3,5	6,5	14,7
	c	1,8	1,8	1.8	1,9	2.4	4.4	6.3	13.0
	d	0,038	0,036	0.040	0,13	0,26	0,23	0,19	0,16

The results indicate that the accuracy increases significantly as one passes from the piecewise-constant approximation to the second-order approximation. Any further improvement in the accuracy as the order of the polynomial is increased to four no longer manifests itself so strongly (case c). But even for the second-order approximation for the internal elements, the inclusion of the asymptotic behaviour of the function being sought over the boundary elements introduces a very substantial contribution (case d). In this case the error is reduced by two orders of magnitude.

In cases a, b, and c, only the central elements are affected as the total number of nodal points is increased to 30. Near the ends the errors remain almost unchanged. Only in case c (taking the asymptotic behaviour into account) the errors are significantly reduced compared with the case of 15 nodal points, namely by a factor of three for the central elements of the crack, by a half for $x = 0.7$, and by 13% for $x = 0.9$.

It follows from these results and from a number of other analogous numerical experiments that increasing the order of the approximation and the total number of nodes alone does not ensure that there will be any increase in accuracy near the ends of the contour. Only the use of special boundary elements that take into account the asymptotic behaviour of the function provides the means for considerably reducing the errors. The introduction of such elements also has a positive effect on the accuracy of computations at the points lying far from the ends. This is in complete agreement with the theoretical analysis (/19/, p.488). Moreover, it is quite sufficient to use the aproximation by polynomials of the second order for the internal elements.

REFERENCES

1. HADAMARD J., The Cauchy Problem for Hyperbolic Linear Partial Differential Equations, Nauka, Moscow, 1978.
2. KANAUN S.K., On the problem of a three-dimensional crack in an anisotropic medium, Prikl. Mat. Mekh., 45, 2, 1981.
3. IOAKIMIDIS N.I., Application of finite-part integrals to the singular integral equations of crack problems in plane and three-dimensional elasticity, Acta Mech., 45, 1-2, 1982.
4. PETUKHOV I.M., ZUBKOV V.V. ZUBKOVA I. A. et al., The state of stress of a rock massif near planned excavation works of arbitrary form, Fiz. Tekh., Probl. Rozrab. Poleznykh Iskopaemykh, 5, 1982.
5. LAZAREV M.I. and SKOVORODA A.R., Determination of elastic stresses by the potential method in three-dimensional case, Izv. Akad. Nauk SSSR MTT, 5, 1983.
6. BELOTSERKOVSKII S.M. and LIFANOV I.K., Numerical Methods in Singular Integral Equations and their Application to Aerodynamics, the Theory of Elasticity, and Electrodynamics, Nauka, Moscow, 1985.
7. UGODCHIKOV A.G. and KHUTORYANSKII N.M., The Method of Boundary Elements in the Mechanics of a Deformable Solid, Izd. Kazan. Univ., Kazan, 1986.
8. LIN'KOV A.M. and MOGILEVSKAYA S.G., Finite-part integrals in three-dimensional crack problems, Prikl. Mat. Mekh., 50, 5, 1986.
9. CROUCH S.L. and STARFIELD A.M., Boundary Element Methods in Solid Mechanics, Mir, Moscow, 1987.
10. WILLMENT T. and BHATTACHARYYA P.K., Displacement discontinuity method with quadratic variation elements, BETECH'86: Proc. 2nd Boundary Elem. Techol. Conf. Mass. Inst. Technol. Southampton, 1986.

11. CURRAN J.H., Numerical modelling of the stress and displacement fields around two- and three-dimensional cracks, Rock mechanics in production and protection, Proc. 25th Sympos. Rock Mech. Evangton 1987, New York, 1987.
12. IOAKIMIDIS N.I., On the validity of the singular integral equations of crack problems at the crack tips, Acta Mech., 48, 3-4, 1983.
13. KIT G.S. and KHAI M.V., Integral equations in three-dimensional problems of thermoelasticity for bodies with cracks, Dokl. Akad. Nauk Ukrain. SSR Ser. A, 12, 1975.
14. KUPRADZE V.D., GEGELIA T.G., BASHELEISHVILI M.O. and BURCHULADZE T.V., Three-dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity, Nauka, Moscow, 1976.
15. ESKIN G.I., Boundary Value Problems of Elliptic Pseudodifferential Equations, Nauka, Moscow, 1973.
16. MUSKHELISHVILI N.I., Singular Integral Equations, Nauka, Moscow, 1968.
17. LIN'KOV A.M.. Integral equations of the theory of elasticity for a plane with cuts loaded with balanced systems of forces, Dokl. Akad. Nauk SSSR, 218, 6, 1974.
18. MUSKHELISHVILI N.I., Some Fundamental Problems of the Mathematical Theory of Elasticity, Nauka, Moscow, 1966.
19. COSTABEL M. and STEPHAN E.P., An imporved boundary elements Galerkin method for three-dimensional crack problems, Integral Equations and Operator Theory, 10, 1987.

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ON THE STRESS-STRAIN STATE NEAR A THREE-DIMENSIONAL CRACK IN A TWO-SHEETED SURFACE*

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A limit scheme of a two-sheeted Riemannian surface is used to illustrate special features encountered in the course of the study of the asymptotic form of the stresses and displacements near the edge of a three-dimensional crack. The fundamental first, second and mixed boundary-value problems are formulated on this surface by analogy with the case of a single plane, and are solved explicitly by quadratures by reducing them to a Riemann boundary-value matrix problem with a constant coefficient. The sheets of the surface are in a generalized plane stress state and have, generally speaking, different stress constants and different thicknesses. A scheme for investigating the stress-strain state of another two-sheeted construction different from the Riemannian surface is elucidated briefly.

A real crystal can naturally be interpreted within the framework of the classical theory of elasticity as a set of elastic interacting planes corresponding to the layers of atoms. Various defects and dislocations /1, 2/ connect the similar surfaces, and it is therefore best to use the methods of the theory of elasticity to multisheeted surfaces when dealing with prescribed types of dislocations.

1. Types of constructions. Let E_1, E_2 be homogeneous, elastic, isotropic infinite thin plates with cuts along the same segment $l_j = [a_j, b_j]$ $(j = 1, 2, \ldots, m)$ of the real x-axis. We shall assume that the plate E_k $(k = 1, 2)$ has a thickness of h_k and is characterized by elastic constants μ_k, $\varkappa_k = (3 - \nu_k)/(1 + \nu_k)$, where μ_k is the shear modulus and ν_k is Poisson's

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